On the proper formulation of Maxwell's equations for waveguide modes

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Abstract

The simplistic approach of discretizing the topological (curl) Maxwell equations when solving for the modes of waveguides leads to spurious modes at zero frequency which fail to satisfy the transversality constraint. We describe here a proper formalism along with a gauge choice which leads to formulations which produce neither null-frequency nor null-field modes.

1 Maxwell equations

We assume a time harmonic field with time dependence $e^{-i\omega t}$. Then the Maxwell equations become

$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} \tag{1}$$

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{D} = 0 \tag{4}$$

These equations contain 4 vector unknowns, which along with the constitutive relations, form a complete system of equations. To reduce the number of unknowns we introduce the vector and scalar potentials:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5}$$

$$\mathbf{E} = -\nabla\phi + i\omega\mathbf{A} \tag{6}$$

Then, the Maxwell equations become

$$\nabla \times \frac{1}{\mu}\nabla \times \mathbf{A} = -i\omega\epsilon\left(-\nabla\phi + i\omega\mathbf{A}\right) \tag{7}$$

$$\nabla \times \left(-\nabla\phi + i\omega\mathbf{A}\right) = i\omega\nabla \times \mathbf{A} \tag{8}$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0 \tag{9}$$

$$\nabla \cdot \epsilon\left(-\nabla\phi + i\omega\mathbf{A}\right) = 0 \tag{10}$$

The middle two equations are identities, so we obtain the new set of equations

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{11}$$

$$\mathbf{E} = -\nabla\phi + i\omega\mathbf{A} \tag{12}$$

$$0 = \nabla \times \frac{1}{\mu}\nabla \times \mathbf{A} - \omega\epsilon\left(\omega\mathbf{A} + i\nabla\phi\right) \tag{13}$$

$$0 = \nabla \cdot \epsilon\left(\omega\mathbf{A} + i\nabla\phi\right) \tag{14}$$

The last equation was originally the divergence-free condition on the displacement field, automatically implied by one of the curl equations. However, at zero frequency, the implication does not hold anymore and the equation must be stated for the equation system to be valid at all frequencies.

2 The eigensystem

In waveguide applications, there is a special coordinate direction, call it the z axis with unit vector $\hat{z}$, and all fields are assumed to have a harmonic variation $e^{i\beta z}$. The differential operator can then be separated as

$$\nabla = \nabla_t + i\beta\hat{z} \tag{15}$$

where ∇_t acts only on components transverse to the z direction. We separate out the field components as follows:

$$\mathbf{A} = e^{i\beta z}\left(\mathbf{A}_t + A_z\hat{z}\right) \tag{16}$$

$$\phi = e^{i\beta z}V \tag{17}$$

We further split $\mathbf{A}_t$ using a Helmholtz decomposition as

$$\mathbf{A}_t = \mathbf{A}_g + \mathbf{A}_c = -\nabla_t\psi + \nabla\times(\zeta\hat{z}) \tag{18}$$

where ψ and ζ are scalar functions of transverse coordinates only. Substituting these forms into the Maxwell equations gives

$$\mathbf{B} = e^{i\beta z}\left[-\hat{z}\nabla_t^2\zeta + \hat{z}\times\left(i\beta\nabla\times(\zeta\hat{z}) - \nabla_t\left(i\beta\psi + A_z\right)\right)\right] \tag{19}$$

$$\mathbf{E} = e^{i\beta z}\left[i\omega\nabla\times(\zeta\hat{z}) + i\nabla_t\left(iV-\omega\psi\right) + \left(i\omega\left(i\beta\psi + A_z\right) - \beta\left(iV-\omega\psi\right)\right)\hat{z}\right] \tag{20}$$

$$0 = -\nabla_t\times\left(\frac{1}{\mu}\hat{z}\nabla_t^2\zeta\right) - \beta^2\hat{z}\times\frac{1}{\mu}\nabla_t\zeta - \omega^2\epsilon\nabla\times(\zeta\hat{z}) \tag{21}$$

$$- i\beta\hat{z}\times\frac{1}{\mu}\hat{z}\times\nabla_t\left(i\beta\psi + A_z\right) - \omega\epsilon\nabla_t\left(iV-\omega\psi\right) \tag{22}$$

$$0 = i\beta\hat{z}\cdot\left(\nabla_t\times\frac{1}{\mu}\nabla_t\zeta\right) - \hat{z}\cdot\left(\nabla_t\times\frac{1}{\mu}\hat{z}\times\nabla_t\left(i\beta\psi + A_z\right)\right) \tag{23}$$

$$- \omega^2\epsilon\left(i\beta\psi + A_z\right) - i\beta\omega\epsilon\left(iV-\omega\psi\right) \tag{24}$$

$$0 = \nabla_t\cdot\epsilon\left[\omega\nabla\times(\zeta\hat{z}) + \nabla_t\left(iV-\omega\psi\right)\right] \tag{25}$$

$$+ i\beta\epsilon\left[i\beta\left(iV-\omega\psi\right) + \omega\left(i\beta\psi + A_z\right)\right] \tag{26}$$

Of the three scalar functions V, ψ, and A_z, the physical fields depend on only two linear combinations:

$$f = i\beta\psi + A_z \tag{27}$$

$$g = iV - \omega\psi \tag{28}$$

Making these substitutions, we obtain

$$\mathbf{B} = e^{i\beta z}\left[-\hat{z}\nabla_t^2\zeta + \hat{z}\times\left(i\beta\nabla\times(\zeta\hat{z}) - \nabla_t f\right)\right] \tag{29}$$

$$\mathbf{E} = e^{i\beta z}\left[i\omega\nabla\times(\zeta\hat{z}) + i\nabla_t g + \left(i\omega f - \beta g\right)\hat{z}\right] \tag{30}$$

We may choose any gauge that does not constrain the scalar functions f and g. Due to the β and ω dependence of f and g, we will fix the gauge by setting $\psi = 0$. We can then write down the Maxwell equations for the potentials as

$$\left(\beta^2 \begin{bmatrix} \hat{z} \times \frac{1}{\mu}\nabla_t & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 & i\hat{z} \times \frac{1}{\mu}\hat{z} \times \nabla_t \\ 0 & 0 & -\omega\epsilon \\ -i\hat{z} \cdot \left(\nabla_t \times \frac{1}{\mu}\nabla_t\right) & -\omega\epsilon & 0 \end{bmatrix}\right. \tag{31}$$

$$\left. + \begin{bmatrix} \omega^2\epsilon\nabla \times (\hat{z}\cdot) + \nabla_t \times \left(\frac{1}{\mu}\hat{z}\nabla_t^2\right) & i\omega\epsilon\nabla_t & 0 \\ i\omega\nabla_t \cdot \epsilon\nabla \times (\hat{z}\cdot) & -\nabla_t \cdot \epsilon\nabla_t & 0 \\ 0 & 0 & \omega^2\epsilon + \hat{z} \cdot \left(\nabla_t \times \frac{1}{\mu}\hat{z} \times \nabla_t\right) \end{bmatrix}\right) \begin{bmatrix} \zeta \\ V \\ A_z \end{bmatrix} = 0 \tag{32}$$

The quadratic term is not full rank, thus there exist eigenvalues $\beta = \infty$ corresponding to axial electric field profiles which violate the transversality constraint. We may form a linear eigenvalue problem by making the substitution $A_z \to -i\beta a_z$. The eigenvalue equation then becomes

$$\left(\beta^2 \begin{bmatrix} \hat{z} \times \frac{1}{\mu}\nabla_t & 0 & \hat{z} \times \frac{1}{\mu}\hat{z} \times \nabla_t \\ 0 & \epsilon & i\omega\epsilon \\ \hat{z} \cdot \left(\nabla_t \times \frac{1}{\mu}\nabla_t\right) & -i\omega\epsilon & \omega^2\epsilon + \hat{z} \cdot \left(\nabla_t \times \frac{1}{\mu}\hat{z} \times \nabla_t\right) \end{bmatrix}\right. \tag{33}$$

$$\left. + \begin{bmatrix} \omega^2\epsilon\nabla \times (\hat{z}\cdot) + \nabla_t \times \left(\frac{1}{\mu}\hat{z}\nabla_t^2\right) & i\omega\epsilon\nabla_t & 0 \\ i\omega\nabla_t \cdot \epsilon\nabla \times (\hat{z}\cdot) & -\nabla_t \cdot \epsilon\nabla_t & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} \zeta \\ V \\ a_z \end{bmatrix} \tag{34}$$

$$= \left(\beta^2 B + A\right) u = 0 \tag{35}$$

Notice however that now there exist null-field solutions with $\beta = 0$ and a_z arbitrary. Assuming there exists orthogonality with respect to B, such that $u_i B u_j = 0$ for different modes i and j, then we may enforce orthogonality by requiring $Bu_j = 0$ for any candidate mode u_j.

3 Differential forms and consistent discretization

We assume that the transverse cross section of the waveguide is triangulated. On this simplicial complex, we have the usual de Rham cohomology, with Hodge stars induced by ϵ and μ^{-1}. The primary concern now is assigning to the scalar potentials ζ, V, and a_z proper discrete differential forms. We assume first that E and H are isomorphic to a primal differential 1- and 2-form, respectively. Then, from the defining relations, V and a_z are primal differential 0-forms, while ζ is a dual differential 1-form.