# Constant frequency modes of a slab waveguide

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**Abstract**

We describe the complete set of modes of an asymmetric slab waveguide at a fixed real frequency. We attempt to show that the set of discrete modes (guided and leaky) form a complete basis and formulate a mode matching method.

## 1 Motivation

The traditional discussion of waveguide modes and mode matching is unsatisfactory for the following reasons: 1. The completeness relation for modes deals with radiation modes, which are in some sense nonphysical due to requiring a planewave source at infinity. 2. The traditional complete basis of modes involving radiation modes forms a continuum, and it is not obvious how a finite approximation should select which radiation modes to consider. 3. The guided+radiation modes do not match what is done numerically when computing modes (guided+leaky). We argue here that guided+leaky modes are a complete basis in an unusual sense. Although the leaky modes are non-normalizable in the traditional sense over the entirety of the transverse cross section of the problem (infinite extent), they are normalizable over any finite interval containing the waveguide. We show that the discretely infinite set of guided+leaky modes forms a complete and orthogonal basis for such intervals, and we argue that these modes are both physically meaningful and satisfy the proper causality requirements. We further demonstrate the correspondence with traditional numerical mode solving techniques and formulate a mode matching technique based on this formalism.

## 2 Governing equations

We begin with the time harmonic ($e^{-i\omega t}$) Maxwell equations:

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} \tag{1}$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} \tag{2}$$

We set the origin of the coordinate system at the center of the slab, with thickness $d$ The $x$ axis is normal the surfaces of the slab, as shown in Fig. 1 If there is a component of the mode wavevector in the plane of the slab, we make the $z$-axis parallel to it, such that we only deal with the $xz$-plane. In this case there is a complete decomposition into TE and TM modes where the $y$ component of the $E$ or $H$ field is nonzero. We consider these two cases separately.

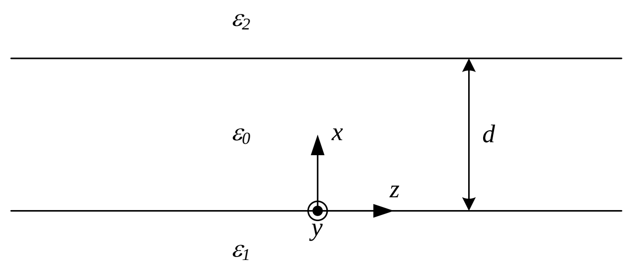


Figure 1: Slab waveguide schematic.

## 3 Eigenvalue problem

Combining the Maxwell equations in a homogeneous region,

$$\nabla \times \nabla \times E_y - \omega^2 \mu \epsilon E_y = -\nabla_{xz}^2 E_y - \omega^2 \mu \epsilon E_y = 0 \tag{3}$$

$$\epsilon \nabla \times \frac{1}{\epsilon} \nabla \times H_y - \omega^2 \mu \epsilon H_y = 0 \tag{4}$$

The general solution is $g(\mathbf{r}) = e^{i\mathbf{k}\cdot\vec{r}}$ where $\mathbf{r} = (x, z)$, and such that

$$\mathbf{k} \cdot \mathbf{k} = \omega^2 \epsilon \mu \tag{5}$$

The general solution can be written in explicit coordinate form as

$$\mathbf{k} = k_x \hat{x} + \beta \hat{z} \tag{6}$$

where

$$k_x^2 + \beta^2 = \omega^2 \mu \epsilon \qquad k_x, \beta \in \mathbb{C} \tag{7}$$

By phase matching, the solution in each region must have an $e^{i\beta z}$ dependence so that the general solution is

$$E_y(x, z) = \begin{cases} C e^{-i\alpha_1 x} e^{i\beta z} & x < 0 \\ \left(A e^{ikx} + B e^{-ikx}\right) e^{i\beta z} & 0 \le x \le d \\ D e^{i\alpha_2 (x-d)} e^{i\beta z} & x > d \end{cases} \tag{8}$$

Therefore, we may collapse the problem into a 1D nonlinear eigenvalue problem. For the TE case,

$$\left[\frac{d^2}{dx^2} + \omega^2 \mu \epsilon_0\right] E_y = \beta^2 E_y \qquad \text{for } 0 < x < d \tag{9}$$

$$\frac{dE_y}{dx} = -i\alpha_1 E_y \qquad \text{at } x = 0 \tag{10}$$

$$\frac{dE_y}{dx} = i\alpha_2 E_y \qquad \text{at } x = d \tag{11}$$

and for the TM case,

$$\left[\epsilon_0 \frac{d}{dx} \frac{1}{\epsilon_0} \frac{d}{dx} + \omega^2 \mu \epsilon_0\right] H_y = \beta^2 H_y \qquad \text{for } 0 < x < d \tag{12}$$

$$\frac{dH_y}{dx} = -i\alpha_1 \frac{\epsilon_0}{\epsilon_1} H_y \qquad \text{at } x = 0 \tag{13}$$

$$\frac{dH_y}{dx} = i\alpha_2 \frac{\epsilon_0}{\epsilon_2} H_y \qquad \text{at } x = d \tag{14}$$

where we define

$$\alpha_i = \sqrt{\omega^2 \mu \epsilon_i - \beta^2} \tag{15}$$

where the square root is the conventional principal square root with positive real part, producing outgoing waves in our formulation.

### 3.1 TE modes

As in [1], we multiply Eq. (9) by $E_y^*$ and integrate over the slab to obtain

$$\int_0^d \left[ \left(\beta^2 - \omega^2 \mu \epsilon_0\right) |E_y|^2 + \left| \frac{dE_y}{dx} \right|^2 \right] dx = i\alpha_1 |E_y(0)|^2 + i\alpha_2 |E_y(d)|^2 \tag{16}$$

where we have integrated by parts and simplified the surface terms using Eqs. (10) and (11). Every quantity of the LHS of Eq. (16) must be real except for $\beta$, while the RHS must lie in the upper half plane by the definition of $\alpha_i$.

The guided modes correspond to real $\beta$, for which the LHS of Eq. (16) must purely real. This forces $\alpha_i$ to be purely imaginary and means that $\beta^2 \geq \omega^2 \mu \epsilon_{1,2}$. Since we also require guided modes to be confined (decay to zero at infinity), we have a strict inequality. The RHS is then strictly negative and forces $\beta^2 < \omega^2 \mu \epsilon_0$. These considerations recover the traditional index guiding conditions.

The complex $\beta^2$ eigenvalues must satisfy $\Im \beta^2 > 0$ since the RHS of Eq. (16) must strictly lie in the upper half plane (not including the real axis). This condition immediately precludes evanescent modes (those which have a decaying exponential dependence in the "propagation" direction) and non-propagating modes (stationary modes for which $\beta = 0$). The conclusion to be drawn from the preceeding two paragraphs is that the set of all possible leaky mode $\beta$ eigenvalues lie *in the first quadrant of the complex plane, including the strictly positive imaginary axis and excluding the real axis.* This observation shows that the $\beta$ corresponding to guided modes lying on the real axis are zeroes on a different branch of the characteristic equation than the zeros of leaky mode $\beta$. The general guiding condition is then

$$\max(\omega^2 \mu \epsilon_1, \omega^2 \mu \epsilon_2) \leq \beta^2 < \omega^2 \mu \epsilon_0 \tag{17}$$

and each $\omega^2 \mu \epsilon_i$ corresponds to a branch point of the characteristic equation on the real axis. Thus guided modes exist only in structures where all $\epsilon_i$ are purely real, although they may be negative. Furthermore, we must have $\epsilon_0 > 0$, but there is no positivity requirement for $\epsilon_1$ and $\epsilon_2$.

### 3.2 TM modes

$$\int_0^d \left[ \left( \frac{\beta^2}{\epsilon_0} - \omega^2 \mu \right) |E_y|^2 + \frac{1}{\epsilon_0} \left| \frac{dE_y}{dx} \right|^2 \right] dx = i \frac{\alpha_1}{\epsilon_1} |E_y(0)|^2 + i \frac{\alpha_2}{\epsilon_2} |E_y(d)|^2 \tag{18}$$

The TM modes show a much richer set of behavior since each term in Eq. (18) can be complex. We still consider $\beta$ in the same region of the first quadrant of the complex plane for leaky modes due to causality.

Guided TM modes require purely imaginary $\alpha_i$ and real $\epsilon_i$.

If $\epsilon_1, \epsilon_2 > 0$, the RHS of Eq. (18) is negative. If $\epsilon_0 > 0$, then guided modes exist for $\beta^2 < \omega^2 \mu \epsilon_0$. Otherwise if $\epsilon_0 < 0$, then guided modes exist for any positive $\beta$.

If $\epsilon_1, \epsilon_2 < 0$, the RHS of Eq. (18) is positive. If $\epsilon_0 > 0$, then guided modes exist for

We now simplify the eigenvalue problem using the general solution. Matching tangential electric and magnetic fields, we arrive at

$$A + B = C \tag{19}$$
$$Ae^{ikd} + Be^{-ikd} = D \tag{20}$$
$$ik\,(A - B) = -i\alpha_1 C \tag{21}$$
$$ik\left(Ae^{ikd} - Be^{-ikd}\right) = i\alpha_2 D \tag{22}$$

Eliminating $C$ and $D$,

$$k\,(A - B) = -\alpha_1\,(A + B)$$
$$k\left(Ae^{ikd} - Be^{-ikd}\right) = \alpha_2\left(Ae^{ikd} + Be^{-ikd}\right)$$

From which we obtain the eigenvalue equation

$$e^{2ikd} = \frac{(k + \alpha_1)\,(k + \alpha_2)}{(k - \alpha_1)\,(k - \alpha_2)} \tag{23}$$

### 3.3 Guided modes

Guided modes are those where $\beta \in \mathbb{R}$ and $\alpha_i$ are purely real. In this case, the RHS of Eq. (23) is exactly unitary, and solutions can exist. To obtain a more traditional eigenvalue equation, we can write

$$2kh + m\pi = \tan^{-1}\frac{\Im\alpha_1}{k} + \tan^{-1}\frac{\Im\alpha_2}{k} \tag{24}$$

For the principal branch of the arctangent, the integer $m$ indexes the guided modes starting from $m = 0$. Since we expect $\Re\alpha_i = 0$, let us denote $\kappa_i = \Im\alpha_i > 0$. From the dispersion relations,

$$k^2 + \beta^2 = \omega^2\mu\epsilon_0 \tag{25}$$
$$-\kappa^2 + \beta^2 = \omega^2\mu\epsilon_i \tag{26}$$

Subtracting gives

$$k^2 + \kappa_i^2 = \omega^2\mu(\epsilon_0 - \epsilon_i) \tag{27}$$

#### 3.3.1 Symmetric case

We write the eigenvalue equation as

$$\tan\left(kh + m\frac{\pi}{2}\right) = \frac{\alpha_0}{k} \tag{28}$$

It is necessary, of course, to also satisfy the dispersion relations

$$k^2 + \beta^2 = \omega^2\mu\epsilon_1 \tag{29}$$
$$-\alpha_0^2 + \beta^2 = \omega^2\mu\epsilon_0 \tag{30}$$

Eliminating $\beta$,

$$k^2 + \alpha_0^2 = \omega^2\mu(\epsilon_1 - \epsilon_0) \tag{31}$$

We can re-arrange this equation to isolate $\alpha_0$:

$$\frac{\alpha_0}{k} = \sqrt{\frac{\omega^2\mu(\epsilon_1 - \epsilon_0)}{k^2} - 1} \tag{32}$$

The textbook treatment of guided modes suggests to simultaneous plot Eqs. (28) and (32) using axes of $\alpha_0/k$ and $kh$ to find solutions $kh$. As long as $\epsilon_1 - \epsilon_0 > 0$, there is at least one guided mode for $m = 0$ (which is not the case for an asymmetric guide).

The maximum value of $k$ is $k_{\text{max}} = \omega\sqrt{\mu(\epsilon_1 - \epsilon)}$, and therefore number of modes is

$$N = \left\lfloor \frac{k_{\text{max}} h}{\pi/2} \right\rfloor + 1 \tag{33}$$

The $m$-th guided mode (indexed starting from 0) must be in the interval $m\frac{\pi}{2} \le k_m h < (m+1)\frac{\pi}{2}$. These bounds allow for simple and efficient numerical computation of the guided modes.

### 3.3.2 Normalization

Guided modes are traditionally normalized to unit power, defined as

$$S = \frac{1}{2}\Re\left(\mathbf{E} \times \mathbf{H}^*\right) \cdot \hat{z} \tag{34}$$

For the TE polarization, we are only interested in $H_x = \frac{i}{\omega\mu}\frac{\partial E_y}{\partial z}$. The power in the mode is then

$$P = \frac{\beta}{2\omega\mu}\int_{-\infty}^{\infty} |E_y|^2 \, dx \tag{35}$$

## 3.4 Leaky modes

Leaky modes are the most difficult case to consider, since they correspond to generally complex $\alpha_i$ and $\beta$. Much of existing literature considers leaky modes (also called leaky waves, PML modes, among other names) to be unphysical, citing the exponentially divergent behavior as $|x| \to \infty$. However a careful analysis reveals that this divergence is caused by the angled phase fronts of the leaking planewaves in the outer regions. The planewaves propagate with a directional component along the waveguide, but also with a directional component away from the waveguide. One may roughly think of the exponential divergence at a particular $z_0$ as due to the leakage from past times (or from fields from $z < z_0$) when the field in the waveguide was exponentially larger.

In [1], asymptotic bounds were given for $k$ in the limit that $|k|$ is large. Computationally, these estimates provide a good initial guess for an iterative numerical method to find the roots of Eq. (23). Although the numerical understanding of how to solve the eigenvalue equation is far less developed than in the guided mode case, the current state-of-the-art is quite sufficient for efficiently computing the leaky modes.

### 3.4.1 Normalization

In contrast with the guided modes, leaky waves do not have a constant power flux in the propagation direction due to leakage. Plotting the Poynting vector density in Fig. 2, we see that in order to normalize the power in the waveguide in a causal way, we should integrate the propagating power only over the slab. In direct analogy with the guided mode case,

$$P = \frac{\Re\beta}{2\omega\mu}\int_{0}^{d} |E_y|^2 \, dx \tag{36}$$

If we do not take the real part, we expect the imaginary part of the power to be related to the outward loss from the slab, but we will not expand on this.

We may also calculate the unit dissipation per unit propagation length of a leaky mode. Adding contributions from leakage on both sides of the slab,

$$S_{\text{tot}} = \frac{1}{2}\Re\left[-E_y(0)H_z(0)^* + E_y(d)H_z(d)^*\right] = \frac{1}{2\omega\mu}\Re\left(\alpha_1\left|E_y(0)\right|^2 + \alpha_2\left|E_y(d)\right|^2\right) \tag{37}$$

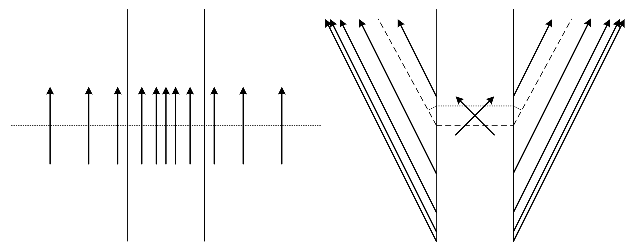

Figure 2: Poynting vector density for a guided mode (left) and leaky mode (right). The arrows point along the Poynting vector direction and follow lines of constant Poynting vector magnitude. The spacing represents the relative magnitude of the Poynting vector in a region. The fine dashed line represents the contour used for normalization of each mode, while the long dashed line represents a particular causal boundary.

## 3.5 Mode orthogonality

From the discussion of normalization, it appears modes have two relevant quantifiers: the propagating power and the amount of leakage per propagation length. Any inner product we define between modes must consider both these quantities.

Generalizing to arbitrary field patterns, an inner product should integrate over only the slab. It must consider the $z$ derivative of the field patterns, as well as the $x$ derivative of the fields just outside the slab.

## 3.6 Completeness of modes

The study of the leaky modes is largely motivated by the interest in understanding coupling to waveguides. The study of modal solutions in the history of physics was primarily motivated by simplifying time evolution in initial value problems; an initial field pattern can be decomposed into modes, attaching to each of a respective time harmonic evolution dependent on the modal eigenvalue, and then superimposing the separate modes. We wish to develop a similar formalism here for waveguides, except at a constant-frequency, we wish to predict the spatial evolution of an initial field pattern specified at $z_0$.

We begin by posing the initial value problem of interest. For the slab waveguide, we seek a solution to

$$\left(\nabla^2_{xz} + \omega^2\mu\epsilon\right)E_y(x,z) = 0 \qquad \text{for } z > z_0 \tag{38}$$

$$E_y(x,z) = g(x) \qquad \text{at } z = z_0 \tag{39}$$

We assume there exists a set of modes $f_n(x)$ with associated $\beta_n$ such that

$$E_y(x, z = z_0) = \sum_n a_n f_n(x) \tag{40}$$

$$\frac{\partial E_y}{\partial z}(x, z = z_0) = \hat{E}_y = \sum_n i\beta_n a_n f_n(x) \tag{41}$$

Note that we have chosen to simultaneously expand both the field value in Eq. (40) as well as its derivative in the propagation direction in Eq. (41). As was noted in [2], this two component expansion is required for a unique expansion because Eq. (38) is second order and thus $E_y(x, z = z_0)$ and $\partial E_y(x, z = z_0)/\partial z$ are arbitrary and can be independently specified.

### 3.7 The adjoint problem

In the Sturm-Liouville theory,

The adjoint problem is the time reversed problem, where all wavevectors are conjugated:

$$\left[\frac{d^2}{dx^2} + \omega^2 \mu \epsilon_0\right] E_y = \overline{\beta}^2 E_y \qquad \text{for } 0 < x < d \tag{42}$$

$$\frac{dE_y}{dx} = -i\overline{\alpha_1} E_y \qquad \text{at } x = 0 \tag{43}$$

$$\frac{dE_y}{dx} = i\overline{\alpha_2} E_y \qquad \text{at } x = d \tag{44}$$

The natural inner product is then

$$\langle f|\, g\rangle = \int_{-\infty}^{\infty} f^* g \, dx \tag{45}$$

where $f^*$ is the adjoint (time-reversal) of $f$.

Our conjecture is that the set of discrete modes (guided and leaky) form a complete basis in the truncated space of a finite interval containing the slab, for functions supported on the interval (possibly requiring knowledge of its $z$ derivative). The conjecture makes sense from causality point of view, since knowledge of the field values outside the interval requires noncausal knowledge of function values outside the interval.

The reason for this suspicion is that in numerical mode solving with finite differences on a PML grid, the computed modes are precisely the guided and leaky modes. In the numerical scheme, these modes are known to form a complete basis (for the vector space of discretized spatial field values). However, the spatial domain is analytically continued into the complex plane, so that it is unclear what the field values in the PML regions should represent.

## References

[1] Jianxin Zhu and Ya Yan Lu. Leaky modes of slab waveguides-asymptotic solutions. *Lightwave Technology, Journal of*, 24(3):1619 –1623, 2006.

[2] P T Leung, S S Tong, and K Young. Two-component eigenfunction expansion for open systems described by the wave equation i: completeness of expansion. *Journal of Physics A: Mathematical and General*, 30(6):2139, 1997.