Optical properties of a dielectric slab

Victor Liu

November 15, 2010

1 Normal incidence response

We assume that we have the following electric fields which are all oriented along the x axis:

$$\begin{aligned}
E_i(z,t) &= E_0 e^{i(kz-\omega t)} && \text{incident} \\
E_r(z,t) &= rE_0 e^{i(-kz-\omega t)} && \text{reflected} \\
E_+(z,t) &= aE_0 e^{i(k_s z-\omega t)} && \text{forward in slab} \\
E_-(z,t) &= bE_0 e^{i(-k_s z-\omega t)} && \text{backwards in slab} \\
E_t(z,t) &= tE_0 e^{i(k(z-d)-\omega t)} && \text{transmitted}
\end{aligned}$$

We assume the slab extends from $z = 0$ to $z = d$. The associated magnetic fields can be found using $\nabla \times E = i\omega\mu H$:

$$\begin{aligned}
H_i(z,t) &= \frac{k}{\omega\mu_0} E_0 e^{i(kz-\omega t)} \\
H_r(z,t) &= r\frac{-k}{\omega\mu_0} E_0 e^{i(-kz-\omega t)} \\
H_+(z,t) &= a\frac{k_s}{\omega\mu_0} E_0 e^{i(k_s z-\omega t)} \\
H_-(z,t) &= b\frac{-k_s}{\omega\mu_0} E_0 e^{i(-k_s z-\omega t)} \\
H_t(z,t) &= t\frac{k}{\omega\mu_0} E_0 e^{i(k(z-d)-\omega t)}
\end{aligned}$$

and are all oriented along the y axis. Matching fields at the interfaces gives

$$\begin{aligned}
1 + r &= a + b & 1 - r &= \frac{k_s}{k}(a - b) \\
ae^{ik_s d} + be^{-ik_s d} &= t & \frac{k_s}{k}\left(ae^{ik_s d} - be^{-ik_s d}\right) &= t
\end{aligned}$$

Solving these gives

$$\begin{aligned}
a &= \frac{2k(k_s + k)e^{-ik_s d}}{(k + k_s)^2 e^{-ik_s d} - (k - k_s)^2 e^{ik_s d}} & b &= \frac{2k(k_s - k)e^{ik_s d}}{(k + k_s)^2 e^{-ik_s d} - (k - k_s)^2 e^{ik_s d}} \\
r &= \frac{(k - k_s)(k + k_s)\sin(k_s d)}{2ikk_s\cos(k_s d) + (k^2 + k_s^2)\sin(k_s d)} & t &= \frac{2ikk_s}{2ikk_s\cos(k_s d) + (k^2 + k_s^2)\sin(k_s d)}
\end{aligned}$$

1.1 Lossless dielectric

For simple lossless dielectrics, $k_s/k = n$, so the power reflection and transmission are given by

$$T = \frac{1}{1 + \frac{(n^2-1)^2}{4n^2}\sin^2(nkd)} \qquad R = 1 - T$$

1.2 Quality factor

The ratio of peak separation to the full width of transmission peaks is

$$\mathcal{F} = \frac{\pi\sqrt{F}}{2} \qquad F = \left(\frac{2r}{1-r^2}\right)^2$$

In the limit of $r \to 1$, we can approximate this as

$$\mathcal{F} = \frac{\pi r}{1-r^2} \approx -\frac{\pi}{2\ln r}$$

Therefore the Q factor of the m-th resonance is

$$Q_m = -\frac{m\pi}{2\ln r}$$

1.2.1 Lorentzian representation

Let

$$L(\delta;\gamma) = \frac{\gamma}{\pi(\delta^2+\gamma^2)} = \Im\frac{1}{\pi(\delta - i\gamma)}$$

Let $\delta = 2nkd$, then we can just as well write

$$T = \frac{2\pi(1-R)^2}{1-R^2} L(\delta;\gamma) * D_{2\pi}(\delta)$$

where $D_{2\pi}(\delta)$ is a Dirac delta comb with period 2π, and $\gamma = -\ln R$, where $R = r^2$ is the power reflectivity from one surface.

1.3 Scattering matrix

Assuming forward and backward wave amplitudes a_1 and b_1 on the left, and a_2 and b_2 on the right, the scattering matrix (assuming a reciprocal slab) is

$$S = \begin{bmatrix} r & t \\ t & r \end{bmatrix}$$

But in general the complex phase of each element depends on the choice of the phase origin for each set of waves (see Haus Ch. 3 for details on reference planes). The element t has poles at

$$nkd = i\ln\left(\pm\frac{n-1}{n+1}\right) = m\pi - i\ln\frac{n-1}{n+1} = m\pi - i\ln r$$

where m is any integer. From above, if there is a pole in the transmission function at z_0, then the quality factor is given by $Q = \Re z_0/(2\Im z_0)$.

1.4 Energy density

The energy density per unit area in the slab is

$$\int_0^d \epsilon E^* E + \mu H^* H dz$$

Focusing on just the electric field component,

$$\begin{aligned} E^* E &= |E_0|^2 \left| a e^{i(k_s z - \omega t)} + b e^{i(-k_s z - \omega t)} \right|^2 \\ &= |E_0|^2 \left[|a|^2 + |b|^2 + 2\Re\left(b^* a e^{2ik_s z} \right) \right] \end{aligned}$$

$$U_e = \int_0^d \epsilon E^* E dz = \epsilon |E_0|^2 \left[\left(|a|^2 + |b|^2 \right) d + 2 \frac{\sin k_s d}{k_s} \Re \left(b^* a e^{ik_s z} \right) \right]$$

For a dielectric slab with index n, and $r = (n-1)/(n+1)$,

$$a = \frac{\frac{2}{n+1}}{1 - \left(\frac{1-n}{1+n}\right)^2 e^{2inkd}} = \frac{(1-r)}{1 - r^2 e^{2inkd}} \qquad b = \frac{\frac{2}{n+1}\frac{n-1}{n+1} e^{2inkd}}{1 - \left(\frac{1-n}{1+n}\right)^2 e^{2inkd}} = \frac{r(1-r)e^{2inkd}}{1 - r^2 e^{2inkd}}$$

$$|a| = \frac{(1-r)^2}{1 + r^4 - 2r^2 \cos 2nkd} \qquad |b| = \frac{r^2(1-r)^2}{1 + r^4 - 2r^2 \cos 2nkd}$$

$$b^* a = \frac{r(1-r)^2 e^{-2inkd}}{1 + r^4 - 2r^2 \cos 2nkd}$$

Then

$$U_e = \epsilon |E_0|^2 \left[\frac{(1+r^2)(1-r)^2 d}{1 + r^4 - 2r^2 \cos 2nkd} + 2 \frac{\sin nkd}{nk} \frac{r(1-r)^2 \cos nkd}{1 + r^4 - 2r^2 \cos 2nkd} \right]$$

$$U_e = \frac{\epsilon |E_0|^2 (1-r)^2 d}{1 + r^4 - 2r^2 \cos 2nkd} \left[(1+r^2) + 2 \frac{\sin nkd}{nkd} r \cos nkd \right]$$

2 Non-normal incidence response

The case of off-normal incidence is very similar to the normal incidence case, except that part of the incident wavevector magnitude is “consumed” by the component parallel to the slab. We choose the y axis to be the direction along which the system and incident planewave is invariant. Therefore $\omega^2 \mu\epsilon = k_x^2 + k_z^2$.

2.1 TE polarization

We assume that the electric field is directed along the y axis, and the magnetic field lies in the xz plane. We can treat only the electric field since the magnetic field can be derived from it.

$$E_i(x,z,t) = E_0 e^{i(k_z z + k_x x - \omega t)} \qquad \text{incident}$$

$$E_r(x,z,t) = r E_0 e^{i(-k_z z + k_x x - \omega t)} \qquad \text{reflected}$$

$$E_+(x,z,t) = a E_0 e^{i(k_s z + k_x x - \omega t)} \qquad \text{forward in slab}$$

$$E_-(x,z,t) = b E_0 e^{i(-k_s z + k_x x - \omega t)} \qquad \text{backwards in slab}$$

$$E_t(x,z,t) = t E_0 e^{i(k(z-d) + k_x x - \omega t)} \qquad \text{transmitted}$$

Note that the phase matching condition requires the k-vector component parallel to the slab be conserved. The associated magnetic fields can be found using $\nabla \times E = i\omega\mu H$:

$$
\begin{aligned}
H_i(z,t) &= (-k_z\hat{x} + k_x\hat{z})\,\frac{1}{\omega\mu_0}E_0 e^{i(k_z z + k_x x - \omega t)} \\
H_r(z,t) &= r\,(k_z\hat{x} + k_x\hat{z})\,\frac{1}{\omega\mu_0}E_0 e^{i(-k_z z + k_x x - \omega t)} \\
H_+(z,t) &= a\,(-k_s\hat{x} + k_x\hat{z})\,\frac{1}{\omega\mu_0}E_0 e^{i(k_s z + k_x x - \omega t)} \\
H_-(z,t) &= b\,(k_s\hat{x} + k_x\hat{z})\,\frac{1}{\omega\mu_0}E_0 e^{i(-k_s z + k_x x - \omega t)} \\
H_t(z,t) &= t\,(-k_z\hat{x} + k_x\hat{z})\,\frac{1}{\omega\mu_0}E_0 e^{i(k(z-d) + k_x x - \omega t)}
\end{aligned}
$$

Matching parallel fields at the interfaces gives

$$
\begin{aligned}
1 + r &= a + b \qquad & 1 - r &= \frac{k_s}{k_z}(a - b) \\
ae^{ik_s d} + be^{-ik_s d} &= t \qquad & \frac{k_s}{k_z}\left(ae^{ik_s d} - be^{-ik_s d}\right) &= t
\end{aligned}
$$

which are identical to the equations in the normal incidence case, except that k is replaced by k_z, leading to an angle dependent reflectivity.

2.2 TM polarization

We assume that the magnetic field is directed along the y axis, and the electric field lies in the xz plane. We can treat only the magnetic field since the electric field can be derived from it. Until now, we have not had to reference the dielectric constants of any materials. We will assume that the slab has $\epsilon = \epsilon_s$. Outside the slab, we assume $\epsilon = \epsilon_0$ (not necessarily vacuum permittivity).

$$
\begin{aligned}
H_i(x,z,t) &= H_0 e^{i(k_z z + k_x x - \omega t)} && \text{incident} \\
H_r(x,z,t) &= rH_0 e^{i(-k_z z + k_x x - \omega t)} && \text{reflected} \\
H_+(x,z,t) &= aH_0 e^{i(k_s z + k_x x - \omega t)} && \text{forward in slab} \\
H_-(x,z,t) &= bH_0 e^{i(-k_s z + k_x x - \omega t)} && \text{backwards in slab} \\
H_t(x,z,t) &= tH_0 e^{i(k(z-d) + k_x x - \omega t)} && \text{transmitted}
\end{aligned}
$$

Note that the phase matching condition requires the k-vector component parallel to the slab be conserved. The associated electric fields can be found using $\nabla \times H = -i\omega\epsilon E$:

$$
\begin{aligned}
H_i(z,t) &= (-k_z\hat{x} + k_x\hat{z})\,\frac{1}{\omega\epsilon_0}E_0 e^{i(k_z z + k_x x - \omega t)} \\
H_r(z,t) &= r\,(k_z\hat{x} + k_x\hat{z})\,\frac{1}{\omega\epsilon_0}E_0 e^{i(-k_z z + k_x x - \omega t)} \\
H_+(z,t) &= a\,(-k_s\hat{x} + k_x\hat{z})\,\frac{1}{\omega\epsilon_s}E_0 e^{i(k_s z + k_x x - \omega t)} \\
H_-(z,t) &= b\,(k_s\hat{x} + k_x\hat{z})\,\frac{1}{\omega\epsilon_s}E_0 e^{i(-k_s z + k_x x - \omega t)} \\
H_t(z,t) &= t\,(-k_z\hat{x} + k_x\hat{z})\,\frac{1}{\omega\epsilon_0}E_0 e^{i(k(z-d) + k_x x - \omega t)}
\end{aligned}
$$

Note that this case is identical to the above, except that instances of k_z and k_s are replaced by k_z/ϵ_0 and k_s/ϵ_s. Whereas $k_s = nk_z$ in the lossless case, here we have $k_s/\epsilon_s = \frac{1}{n}k_z/\epsilon_0$.

3 Modal structure

Here we consider the modal structure of the slab. We consider modes to be field patterns which persist in time with a particular constant spatial shape, but it can be decaying in time. Therefore, we consider not only spatially stationary oscillation field patterns at a real-valued frequency, but also modes at complex frequency with negative imaginary part (in order to have exponential decay since our time convention is $e^{-i\omega t}$).

Since there is a mirror symmetry about the center of the slab, we have a decomposition into even and odd modes. We thus only need to consider the half-infinite space on one side of the center of the slab. Placing the origin at the center of the slab, we denote the half-thickness of the slab by a (in the positive z direction), with the same coordinate system as above. We will refer to the half-space outside the half-slab as region 0, and the region inside the half-slab as region 1.

The modes in a uniform space are simply the planewaves, with variation $\exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$, with dispersion relation $\omega^2 = v_p^2\,|k|^2$ (which forbids non-real ω).

We consider here the 1D case first, in which we have a slab of index n and half-thickness a, for which we know there exist even and odd modes, and hence need only consider a halfspace $z > 0$. We use the quasi-normal mode formalism, in which we form two test functions which individually satisfy either the boundary condition at $z = 0$ or $z = a$. The test functions we consider are

$$
\begin{aligned}
f(k,z) &= \left\{\begin{array}{c}\cos\\ \sin\end{array}\right\}(kz) && z < a \\
g(k,z) &= e^{ikz} && z > a \\
g(k,z) &= \alpha\sin(nkz) + \beta\cos(nkz) && z < a
\end{aligned}
$$

where

$$
\alpha = \frac{e^{ika}}{n}(i\cos(nkz) + n\sin(nkz)) \qquad \beta = \frac{e^{ika}}{n}(n\cos(nkz) - i\sin(nkz))
$$

The quasi-normal modes are given by the zeros of the Wronskian of f and g; it is $k\alpha$ in the even case (for which $k = 0$ is a true residue) and $k\beta$ in the odd case (for which $k = 0$ is not a true residue). Rewriting α and β, the QNM modes are equivalently the roots of

$$ne^{iak-iank} - ne^{iank+iak} + e^{iak-iank} + e^{iank+iak}$$

$$ne^{iak-iank} + ne^{iank+iak} + e^{iak-iank} - e^{iank+iak}$$

Solving these gives

$$k = -\frac{i}{na} \ln \sqrt{\frac{n+1}{n-1}} + \text{FSR}$$

which is identical to the poles of the scattering matrix computed above, and correctly gives the real frequency and quality factors of each resonance.