Maxwell's equations

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Abstract

This is a draft only. Please inform me of any errors, points needing clarification, organizational changes, etc. Particularly important are assumptions I may have forgotten to state explicitly.

Contents

1 Definition

1.1 Pure form

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

where ρ and $\boldsymbol{J}$ are total charge density and total current density (bound and free). All fields are assumed to be functions of space and time.

1.2 Material form

Alternatively, we separate out the material properties by setting

$$\rho = \rho_b + \rho_f \qquad \boldsymbol{J} = \boldsymbol{J}_b + \boldsymbol{J}_f$$
$$\rho_b = -\nabla \cdot \boldsymbol{P} \qquad \boldsymbol{J}_b = \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}$$
$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \qquad \boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$$

Material property definitions

so then Maxwell's equations become

$$\nabla \cdot \boldsymbol{D} = \rho_f \qquad \nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t}$$

Free/bound charge formulation

The set of 10 equations listed thus far contains 12 variables, and hence is not closed. Two additional constitutive relations must be specified:

$$\boldsymbol{P}(\boldsymbol{r}, t) = \epsilon_0 \int d^3\boldsymbol{r}' dt' \chi_e(\boldsymbol{r}, \boldsymbol{r}', t, t'; \boldsymbol{E}) \boldsymbol{E}(\boldsymbol{r}', t')$$
$$\boldsymbol{M}(\boldsymbol{r}, t) = \frac{1}{\mu_0} \int d^3\boldsymbol{r}' dt' \chi_m(\boldsymbol{r}, \boldsymbol{r}', t, t'; \boldsymbol{B}) \boldsymbol{B}(\boldsymbol{r}', t')$$

most general constitutive relations

It is this form that will be primarily used throughout the rest of this work, since we are most interested in the effect of material properties.

1.3 Special cases of susceptibilities

Typically, the susceptibilities χ_e and χ_m are spatially localized functions, so that they do not depend on position. Mathematically,

$$\chi_e(\boldsymbol{r}, \boldsymbol{r}', t, t'; \boldsymbol{E}) = \chi_e(t, t'; \boldsymbol{E}) \delta(\boldsymbol{r} - \boldsymbol{r}')$$
$$\chi_m(\boldsymbol{r}, \boldsymbol{r}', t, t'; \boldsymbol{B}) = \chi_m(t, t'; \boldsymbol{B}) \delta(\boldsymbol{r} - \boldsymbol{r}')$$

Then the constitutive relations become

$$\boldsymbol{P}(\boldsymbol{r}, t) = \epsilon_0 \int dt' \chi_e(t, t'; \boldsymbol{E}) \boldsymbol{E}(\boldsymbol{r}, t')$$
$$\boldsymbol{M}(\boldsymbol{r}, t) = \frac{1}{\mu_0} \int dt' \chi_m(t, t'; \boldsymbol{B}) \boldsymbol{B}(\boldsymbol{r}, t')$$

no spatial dispersion

Furthermore, if the material is causal, linear, and time-invariant, we may instead write

$$\boldsymbol{P}(\boldsymbol{r},t) = \epsilon_0 \int_{-\infty}^{t} dt' \chi_e(t-t')\boldsymbol{E}(\boldsymbol{r},t') = \epsilon_0 \chi_e(\omega)\boldsymbol{E}(\boldsymbol{r},\omega)$$
$$\boldsymbol{M}(\boldsymbol{r},t) = \frac{1}{\mu_0} \int_{-\infty}^{t} dt' \chi_m(t-t')\boldsymbol{B}(\boldsymbol{r},t') = \frac{1}{\mu_0} \chi_m(\omega)\boldsymbol{B}(\boldsymbol{r},\omega)$$

linearly dispersive material

subject to the Kramer's-Kronic relations for the real and imaginary parts:

$$\Re\{\chi(\omega)\} = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\Im\{\chi(\omega')\}}{\omega'-\omega} d\omega'$$
$$\Im\{\chi(\omega)\} = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\Re\{\chi(\omega')\}}{\omega'-\omega} d\omega'$$

causality constraints for linearly dispersive materials

At this point, we arrive at the typically recited linearly dispersive constitutive relations $\boldsymbol{D} = \epsilon\boldsymbol{E}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$. If we go further and assume the material has instantaneous response, then the material is no longer dispersive. Mathematically, the statement becomes

$$\chi_e(t,t') = \chi_e\delta(t-t') \qquad \chi_m(t,t') = \chi_m\delta(t-t')$$

and so ϵ and μ are not frequency dependent.

1.4 Related quantities

1.4.1 Classical force

The Lorentz force for a single charged particle is

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

particle Lorentz force

For a charge and current density, we generalize this to

$$\boldsymbol{F} = \int_{\Omega} d^3\boldsymbol{r}\rho(\boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B})$$

distribution Lorentz force

From these we may derive the maxwell stress tensor [?]

$$T_{\mu\nu} = \epsilon E_\mu E_\nu + \mu H_\mu H_\nu - \frac{\delta_{\mu\nu}}{2}\left[\boldsymbol{E}\cdot\boldsymbol{E} + \boldsymbol{H}\cdot\boldsymbol{H}\right]$$

Maxwell stress tensor; linear dispersive

from which the force an a general body is

$$\boldsymbol{F} = \int_{\partial\Omega} T_{\mu\nu} d\boldsymbol{a}_\nu$$

force from stress tensor

1.4.2 Poynting vector

The energy density flux carried by the fields is

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

Poynting vector; lossless nondispersive linear

The generalization to dispersive and lossy cases is below.

1.5 Symmetry relations

1.5.1 Continuity equation

1.5.2 Reciprocity

2 Planewave basis

The utility of the Fourier transform technique suggests planewave solutions, requiring the use of complex-valued fields. Since all physical fields are purely real, it is assumed that to obtain a physical field from a complex-valued field, only the real part is taken. Taking the curl of the source free Faraday's law,

$$\nabla \times \nabla \times \boldsymbol{E} + \nabla \times \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

The order of differentiation on the right may be interchanged by relativistic invariance. Here we must assume a linear dispersive medium, so that $\boldsymbol{B} = \mu \boldsymbol{H}$, then using the source-free Ampere-Maxwell law,

$$\nabla \left(\nabla \cdot \boldsymbol{E} \right) - \nabla^2 \boldsymbol{E} + \epsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \left[\nabla^2 - \epsilon\mu \frac{\partial^2}{\partial t^2} \right] \boldsymbol{E} = 0$$

which is a vector wave equation. This suggests that there exist planewave solutions of the form $\boldsymbol{E} = \boldsymbol{E}_0 e^{i\omega t - \boldsymbol{k}\cdot\boldsymbol{r}}$. This is the time-harmonic convention used throughout. Substituting this proposed form of the solution back into Maxwell's equations, we arrive at

$$\begin{aligned} \boldsymbol{k} \times \boldsymbol{E} &= i\omega \boldsymbol{B} \\ \boldsymbol{k} \times \boldsymbol{H} &= -i\omega \boldsymbol{D} \end{aligned}$$

planewave Maxwell's equations, linear dispersive

The above equations assume a uniform linear dispersive medium (so that $\boldsymbol{D}(\omega) = \epsilon(\omega)\boldsymbol{E}(\omega)$, etc.). Note that the divergence relations are no longer needed since planewaves automatically satisfy them.

2.1 Operator theory

Deriving the vector Helmholtz equations from source-free Maxwell's equations for linear dispersive materials in time-harmonic form gives

$$\left[\frac{1}{\epsilon}\nabla\times\left(\frac{1}{\mu}\nabla\times\right)-\omega^2\right]\boldsymbol{E}=0$$
$$\left[\frac{1}{\mu}\nabla\times\left(\frac{1}{\epsilon}\nabla\times\right)-\omega^2\right]\boldsymbol{H}=0$$

time harmonic Helmholtz equations

These define eigenvalue equations with operators

$$\Xi=\frac{1}{\epsilon}\nabla\times\left(\frac{1}{\mu}\nabla\times\right) \qquad \Theta=\frac{1}{\mu}\nabla\times\left(\frac{1}{\epsilon}\nabla\times\right)$$

For the straightforward inner product $\langle\boldsymbol{E}_1,\boldsymbol{E}_2\rangle=\int d^3\boldsymbol{r}\boldsymbol{E}_1\boldsymbol{E}_2$, the adjoint operators are

$$\Xi^*=\nabla\times\left(\frac{1}{\mu^*}\nabla\times\frac{1}{\epsilon^*}\right) \qquad \Theta^*=\nabla\times\left(\frac{1}{\epsilon^*}\nabla\times\frac{1}{\mu^*}\right)$$

3 Solutions to Maxwell's equations

Here we assume all media to be causal, linear, and time-invariant unless otherwise specified. In other words, assume materials are linear dispersive.

3.1 Uniform medium

3.1.1 Planewave solutions

3.1.2 Green's function solutions

3.1.3 Hopf fibration

3.2 Media interface

3.2.1 The Fresnel equations

3.2.2 Total internal reflection

3.3 Periodic media

3.3.1 General principles

3.3.2 Methods of solutions for finite 1D problems

3.3.3 Operator theory

3.4 Pseudoperiodic media

4 Derivations

4.1 Special relativity

4.2 Classical field theory

5 General covariance

6 References