Worked examples with Einstein summation notation

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Abstract

Some examples of using Einstein summation notation are presented. All manipulations assume a three-dimensional space with cartesian coordinates (rather, the basic identities and conversions from vector calculus only work in cartesian coordinates). The point here is to use Einstein notation as a way of getting from one vector identity to another.

1 Basics from vector calculus

Dot product	$\mathbf{u} \cdot \mathbf{v}$	$u_i v_i$
Cross product	$\mathbf{u} \times \mathbf{v}$	$\epsilon_{ijk} u_j v_k$
Gradient	∇f	$\partial_i f$
Divergence	$\nabla \cdot \mathbf{u}$	$\partial_i u_i$
Curl	$\nabla \times \mathbf{u}$	$\epsilon_{ijk} \partial_j v_k$
Directional derivative	$\mathbf{u} \cdot \nabla \mathbf{v}$	$u_i \partial_i v_j$
Vector Laplacian	$\nabla^2 \mathbf{u}$	$\partial_i \partial_i u_j$

2 Identities

Antisymmetry of permutation	$\epsilon_{ijk} = -\epsilon_{jik} = \epsilon_{jki} = \dots$
Derivatives of orthonormal basis	$\partial_i e_j = 0$
Derivatives of coordinates	$\partial_i x_j = \delta_{ij}$
Permutations differing by 1 index	$\epsilon_{jmn} \epsilon_{imn} = 2\delta_{ij}$
Permutations with identical indices	$\epsilon_{ijk} \epsilon_{ijk} = 6$ (twice the dimensionality)
Permutations differing by 2 indices	$\epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$

3 Examples

3.1 A vector identity

Show $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

$$\begin{aligned}
\epsilon_{ijk} a_j (\epsilon_{klm} b_l c_m) &= \epsilon_{kij} \epsilon_{klm} a_j b_l c_m \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m \\
&= a_j b_i c_j - a_j b_j c_i \\
&= (a_j c_j) b_i - (a_j b_j) c_i
\end{aligned}$$

3.2 Proof of vector calculus null identities

Show $\nabla \times (\nabla f) = 0$ and $\nabla \cdot (\nabla \times \mathbf{u}) = 0$.

$$\begin{aligned}
\nabla \times (\nabla f) &= \epsilon_{ijk}\partial_j\partial_k f \\
&= -\epsilon_{ikj}\partial_j\partial_k f \\
&= -\epsilon_{ikj}\partial_k\partial_j f \\
&= -\epsilon_{ijk}\partial_j\partial_k f \\
&= 0
\end{aligned}$$

This is shown by antisymmetry, since we are free to swap the order of differentiation in the third to last line and switch all instances of j and k in the second to last line.

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{u}) &= \partial_i\epsilon_{ijk}\partial_j u_k \\
&= \epsilon_{ijk}\partial_i\partial_j u_k \\
&= -\epsilon_{jik}\partial_i\partial_j u_k \\
&= -\epsilon_{jik}\partial_j\partial_i u_k \\
&= -\epsilon_{ijk}\partial_i\partial_j u_k \\
&= 0
\end{aligned}$$

The proof is analogous to the previous case.

3.3 The vector Laplacian identity

Show $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}$.

$$\begin{aligned}
\epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l u_m) &= \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l u_m \\
&= \epsilon_{kij}\epsilon_{klm}\partial_j\partial_l u_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta jl)\partial_j\partial_l u_m \\
&= \partial_j\partial_i u_j - \partial_j\partial_j u_i \\
&= \partial_i\partial_j u_j - \partial_j\partial_j u_i \\
&= \partial_i(\partial_j u_j) - (\partial_j\partial_j)u_i
\end{aligned}$$

3.4 An example from fluid mechanics

Show

$$\mathbf{v} \cdot \nabla\mathbf{v} = \nabla\left(\frac{|\mathbf{v}|^2}{2}\right) + (\nabla \times \mathbf{v}) \times \mathbf{v}$$

$$\begin{aligned}
v_a\partial_a v_b &= \partial_b\left(\frac{v_a v_a}{2}\right) + \epsilon_{bac}(\epsilon_{adf}\partial_d v_f)v_c \\
&= \frac{1}{2}v_a\partial_b v_a + \frac{1}{2}(\partial_b v_a)v_a - \epsilon_{abc}(\epsilon_{adf}\partial_d v_f)v_c \\
&= v_a\partial_b v_a - (\epsilon_{abc}\epsilon_{adf})(\partial_d v_f)v_c \\
&= v_a\partial_b v_a - (\delta_{bd}\delta_{cf} - \delta_{bf}\delta_{cd})(\partial_d v_f)v_c \\
&= v_a\partial_b v_a - (\partial_b v_c)v_c + (\partial_c v_b)v_c \\
&= v_a\partial_b v_a - v_a(\partial_b v_a) + v_a(\partial_a v_b) \\
&= v_a\partial_a v_b
\end{aligned}$$