Expressing isolated delta functions in terms of the comb function

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Abstract

Identities relating the Dirac delta function and comb function in terms of a high frequency comb function.

1 Convention

The Fourier transform convention is

$$F(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \mathcal{F}^{-1}\{F(k)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

Thus, the Dirac delta function satisfies

$$\mathcal{F}\{\delta(x)\} = 1$$

The comb function of period T is defined as

$$Ш_T(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT)$$

which is related to the unit period comb function by

$$Ш_T(x) = \frac{1}{T} Ш\left(\frac{x}{T}\right)$$

and its Fourier transform is

$$\mathcal{F}\{Ш_T(x)\} = \sum_{n=-\infty}^{\infty} e^{-iknT}$$

with a Fourier series

$$Ш_T(x) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nx/T}$$

Therefore we can conclude that

$$\mathcal{F}\{Ш_T(x)\} = \frac{1}{T} Ш_{1/T}\left(\frac{k}{2\pi}\right)$$

2 Delta function

$$\begin{aligned}
\delta(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dk \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{\frac{2\pi n}{T}-\frac{\pi}{T}}^{\frac{2\pi n}{T}+\frac{\pi}{T}} e^{-ikx}dk \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{-i(k+2\pi n/T)x}dk \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{-ikx}dk\left(\sum_{n=-\infty}^{\infty} e^{-i2\pi nx/T}\right) \\
&= Ш_T(x)\frac{\sin\frac{\pi x}{T}}{\frac{\pi x}{T}}
\end{aligned}$$

One may think of the effect of the sinc term as cancelling off all delta peaks except for the one at zero. Alternatively,

$$\begin{aligned}
\delta(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dk \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{\frac{2\pi n}{T}-\frac{\pi}{T}}^{\frac{2\pi n}{T}+\frac{\pi}{T}} e^{-ikx}dk \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{-i(k+2\pi n/T)x}dk \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}\sum_{n=-\infty}^{\infty} e^{-i(k+2\pi n/T)x}dk \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{-ikx}Ш_T(x)dk
\end{aligned}$$

3 Lower frequency comb function

Assuming $T = Nt$, for integral $N > 0$,

$$
\begin{aligned}
Ш_T(x) &= \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nx/T} \\
&= \frac{1}{Nt} \sum_{n=-\infty}^{\infty} e^{i2\pi nx/(Nt)} \\
&= \frac{1}{Nt} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{N-1} e^{i2\pi(nN+m)x/(Nt)} \\
&= \left(\frac{1}{t} \sum_{n=-\infty}^{\infty} e^{i2\pi nx/t} \right) \left(\frac{1}{N} \sum_{m=0}^{N-1} e^{i2\pi mx/(Nt)} \right) \\
&= \frac{1}{N} \sum_{m=0}^{N-1} e^{i2\pi mx/T} Ш_t(x) \\
&= Ш_t(x) \frac{e^{i2\pi Nx/T} - 1}{e^{i2\pi x/T} - 1} \\
&= Ш_t(x) \frac{e^{i\pi(N-1)x/T}}{N} \frac{\sin \frac{\pi Nx}{T}}{\sin \frac{\pi x}{T}}
\end{aligned}
$$

So then

$$
\begin{aligned}
\mathcal{F}\{Ш_T(x)\} &= \frac{1}{N} \sum_{m=0}^{N-1} \delta\left(\frac{k}{2\pi} - \frac{m}{T} \right) * \left[\frac{1}{t} Ш_{1/t}\left(\frac{k}{2\pi} \right) \right] \\
&= \frac{1}{T} \sum_{m=0}^{N-1} Ш_{1/t}\left(\frac{m}{T} \right)
\end{aligned}
$$

which is a rather trivial result, since it is simply the decimation of a summation into interleaved sets. Alternatively,

$$
\mathcal{F}\{Ш_{Na}(x)\} = \frac{2\pi}{Na} \sum_{m=0}^{N-1} Ш_{2\pi/a}\left(\frac{2\pi m}{Na} \right)
$$