Arc spline reference

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1 Notation

Vector quantities will be denoted with uppercase letters, and scalars with lowercase letters.

2 Representation

We consider first the representation of a single circular arc segment. This representation must be able to represent a circular arc as well as a line segment. We expect a greater need to represent arcs of smaller interior angle than large (large angle arcs can be subdivided into small angle arcs, anyways), so that we choose the following representation, explained in detail below:

$$[A, B, g]$$

The endpoints of the arc or line segment are A and B, while half the chordal distance from A to B is t. The height of the arc from the chord to the midpoint of the arc M is h, d is the distance from the midpoint of the chord to the center C of the circle of radius r, and θ is half the central angle of the arc. The third parameter of the representation is the "bulge" $g = h/t$. Here we will consider the case where $g \ll 1$ as that typically causes the most numerical headaches since the computation of the radius and circle center are ill-conditioned. In the case where g is not diminutively small, computations involving the circle center are suitable.

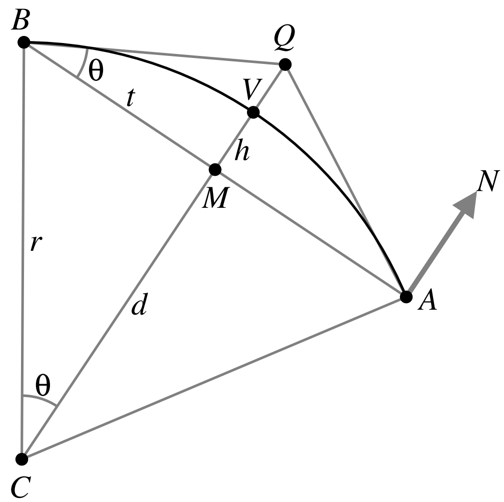

Figure 1: Arc spline representation.

There are a number of useful relations:

$$g = \tan\frac{\theta}{2} \tag{1}$$

$$d = t\frac{1-g^2}{2g} \tag{2}$$

$$r = h + d = t\frac{1+g^2}{2g} \tag{3}$$

$$\cos\theta = \frac{d}{r} = \frac{1-g^2}{1+g^2} \tag{4}$$

$$\sin\theta = \frac{2g}{1+g^2} \tag{5}$$

$$\tan\theta = \frac{t}{d} = \frac{2g}{1-g^2} \tag{6}$$

$$\frac{r\theta}{t} = \text{relative arc length} = \frac{1+g^2}{2g}\theta \tag{7}$$

$$N \equiv \frac{(B-A)^{-\perp}}{\|B-A\|} \tag{8}$$

The bulge parameter g is essentially obtained from the Weierstrass substitution of $\sin\theta$ and $\cos\theta$. For an arc with a bulge parameter g, the arc that is the complement of the arc with respect to its defining circle has bulge parameter $-1/g$.

3 Determination from three points

An alternative parameterization for a circular arc is to specify the endpoints and another point which lies on the arc. This representation requires 6 real numbers instead of 5, and has the disadvantage of being sensitive to the position of the point on the arc when it is near and endpoint. We seek a way to convert from this representation to the convention described above. Let us denote the endpoints by A and B, and the third point on the arc is P.

We will choose the midpoint M as the relative origin, and compute $P - M$. Let us define

$$\alpha = \frac{(P - M) \cdot (M - A)}{\|M - A\|^2}$$
$$\beta = \frac{(P - M) \times (M - A)}{\|M - A\|^2}$$

These are the projections of P onto the segment AB and its perpendicular, but scaled relative to t. With this, we can use the equation of the circle to write

$$(\alpha t)^2 + (\beta t + d)^2 = t^2 + d^2$$
$$(\alpha t)^2 + (\beta t)^2 + 2\beta t d - t^2 = 0$$
$$\beta g^2 + g\left(1 - \alpha^2 - \beta^2\right) - \beta = 0$$

This can be solved for g. Note that there are two roots to the equation, g and $-1/g$, corresponding to the same circle, but complementary arcs. The sign of g must be chosen to match the sign of β.

4 Rational quadratic Bézier spline representation

An arc can be represented exactly by a rational quadratic Bézier spline. The two control points for the ends correspond to A and B both with weight 1. The weight w of the center control point Q is $w = \cos\theta$ and $Q = M + N\frac{2g}{1-g^2}t$. This representation is necessary for robust intersection calculations between arcs and other curves.

In computing intersection points, the implicitized form is often more desirable than the parametric form. Let τ_0, τ_1, and τ_2 be the barycentric coordinates of a point with respect to A, Q, and B, respectively. Then the implicitized curve is $f(x, y) = \tau_1^2 - 4w\tau_0\tau_2$, which is identically zero on the curve itself. The exact curve corresponds, obviously to cases where $\tau_i \geq 0$. In the case of a nearly linear arc, the barycentric coordinates may be numerically difficult to compute. In that case, the equation above for Q should be considered a small perturbation of the point M, and this additional knowledge of the symmetry of the curve may be used. We seek to find the coordinates of a point $X = \lambda_1 A + \lambda_2 B + \lambda_3 Q$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Instead, we write

$$X = \lambda_1 [M + (A - M)] + \lambda_2 [M + (B - M)] + \lambda_3 [M + (Q - M)]$$
$$X - M = \lambda_1 (A - M) + \lambda_2 (B - M) + \lambda_3 (Q - M)$$

Now letting $\lambda_3 = 1 - \lambda_1 - \lambda_2$,

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{T}^{-1} [(X - M) - (Q - M)]$$

where $\mathbf{T}$ is a transformation matrix

$$\mathbf{T} = \begin{bmatrix} (A - M) - (Q - M) & (B - M) - (Q - M) \end{bmatrix}$$

Explicitly,

$$\lambda_1 = \frac{[(X-M)-(Q-M)] \times [(B-M)-(Q-M)]}{\det \mathbf{T}} \tag{9}$$

$$\lambda_2 = \frac{[(X-M)-(Q-M)] \times [(Q-M)-(A-M)]}{\det \mathbf{T}} \tag{10}$$

where $\det \mathbf{T} = \frac{4gt^2}{1-g^2}$.

5 Triangular hull

The control polygon of the rational quadratic Bézier representation always contains the curve within it, but at larger θ, it becomes less tight. The actual triangular hull can be computed with knowledge of θ. The apex point is

$$Q = M + Nt \tan\theta$$

6 Offsetting

The offset curve to an arc segment of distance δ is

$$\text{off}_d\,[A, B, g] = \left[A', B', g\right]$$

where g does not change since the new arc is similar to the original. The endpoints are calculated straightforwardly given knowledge of the chord and central angle:

$$A' = A + \delta \frac{(Q-A)^{-\perp}}{\|Q-A\|} \tag{11}$$

$$B' = B + \delta \frac{(B-Q)^{-\perp}}{\|B-Q\|} \tag{12}$$

With this, the new half-chord distance can be computed: $t' = \frac{1}{2}\,\|B' - A'\|$.

Note that the case of an arc inversion when $g\delta < 0$ and $|\delta| > r$ is also handled correctly.

7 Orientation predicate

Assuming a point X lies within the infinite wedge defined by the arc, determining which side of the arc X lies is simple when C is computable. Clearly, if X is not within the triangular hull of the arc, the question reduces to which side of the chord X lies.

The only difficult case is when $g \ll 1$ and X lies within the triangular hull. In this case, the arc can be repeated bisected until X is no longer within the hull, and a simple test against the chord is performed.

8 Distance queries

It is sometimes necessary to compute the distance from a point X to the nearest point on an arc. This is essentially a refinement of the orientation predicate to also providing distance information. For points outside the circular sector of angle 2θ, the distance is simply one of the distances to an endpoint, so we disregard these cases. The lines defining this sector can be computed with knowledge of the endpoints and θ (indeed, they were computed for the endpoints of an offset curve).

For points within the infinite wedge, if the center and radius were available, the distance would simply be $d = |\|X - C\| - r|$, with the nearest point defined by the intersection of the line from X to C with the arc. For $g \ll 1$, it is not possible to compute C accurately...

Notice first that the distance from X to the endpoints is a local maximum in terms of the arc length parameter. Thus the distance from X to V should be less than the distance to either A or B. We can continue to bisect the arc, each time keeping the half which satisfies this condition. In the limit, the distance from the chord to X is the desired distance.

9 Approximation by cubic Bézier

The two interior control points of the Bézier spline must be arrange symmetrically with respect the middle of the arc, while the endpoints coincide with the endpoints of the arc. Denote by A' and B' the interior control points nearer to A and B, respectively. Since the endpoint tangents should match the arc's tangents, the point A' (and hence also B') are determined uniquely by a single parameter α which is the proportional to the distance from A to A'. We can parameterize the interior control points by

$$A' = A + \alpha V + \beta N$$
$$B' = B - \alpha V + \beta N$$

where

$$V \equiv \frac{B - A}{2}$$

and

$$\beta = \alpha \|V\| \tan\theta$$

The Bézier curve itself is defined by

$$P(t) = (1-t)^3 A + 3t(1-t)^2 A' + 3t^2(1-t)B' + t^3 B$$

We will work in a reference frame relative to M, so that

$$P(t) = M + V(2t-1)\left[1 + t(1-t)(2-3\alpha)\right] + 3\beta t(1-t)N$$

The derivative is then

$$P'(t) = 3V\left[\alpha + 2t(1-t)(2-3\alpha)\right] + 3\beta(1-2t)N$$

In order to minimize the error between the Bézier spline and the arc, we can simply force the midpoint of the spline to coincide with the midpoint of the arc:

$$P\left(\frac{1}{2}\right) = M + hN$$

This reduces to

$$\beta = \frac{4}{3}h \qquad \alpha = \frac{2}{3}\left(1 - g^2\right)$$

To obtain a better approximation we can set the area of the circular cap equal to the area of the cap defined by the spline:

$$r^2\theta - tc = -\int_0^1 (P(t) - M) \times P'(t)dt$$

The integrand simplifies to

$$\begin{aligned}(P(t) - M) \times P'(t) = (V \times U) &\left[-3\beta(2t-1)^2\left(1 + t(1-t)(2-3\alpha)\right)\right.\\ &\left.-\left(3\beta t(1-t)\right)3\left(\alpha + 2t(1-t)(2-3\alpha)\right)\right]\end{aligned}$$

Integrating results in the condition

$$-\int_0^1 (P(t) - M) \times P'(t)dt = \frac{3}{5}\beta\,\|V\|\,(4-\alpha) = \frac{3}{5}\alpha(4-\alpha)\,\|V\|^2\tan\theta$$

Which results in the condition

$$\left(\frac{1+g^2}{2g}\right)^2 \tan^{-1}\frac{2g}{1-g^2} - \frac{1-g^2}{2g} = \frac{3}{5}\alpha(4-\alpha)\frac{2g}{1-g^2}$$

The actual cap area can be expanded as

$$\left(\frac{1+g^2}{2g}\right)^2 \tan^{-1}\frac{2g}{1-g^2} - \frac{1-g^2}{2g} \approx \frac{4g}{3}\left[1 + \frac{g^2}{5} - \frac{g^4}{35} + O(g^6)\right]$$

Linearizing with respect to g gives

$$\frac{4g}{3} \approx \frac{3}{5}\alpha(\alpha-4)2g$$

Which gives $\alpha = 2 - \sqrt{26/9} \approx 0.300$. Keeping quadratic terms gives

$$\alpha = 2 - \sqrt{2\left[1 + \frac{\left(2+g^2\right)^2}{9}\right]}$$

10 Arc length parameterization

The total arc length is just $r\theta = t\frac{\sin^{-1}\alpha}{\alpha}$ where $\alpha = \sin\theta = \frac{2g}{1+g^2}$. This can generally be computed robustly even for small α by appropriate treatment near $\alpha = 0$.

More generally, the arc length parameterization in terms of $s \in [0,1]$ is given by the spherical linear interpolation formula

$$P(s) = C + \frac{\sin 2(1-s)\theta}{\sin 2\theta}(A - C) + \frac{\sin 2s\theta}{\sin 2\theta}(B - C) = C + \alpha(s)(A - C) + \beta(s)(B - C)$$

Once again, since C cannot be relied upon, we must derive a different expression. First let us change to the coordinates $\sigma = s - \frac{1}{2}$ which is symmetric about the center of the arc. We can now write

$$P(s) = \alpha A + \beta B - (\alpha + \beta - 1)C$$

where

$$\alpha(\sigma) = \frac{\sin 2\left(\frac{1}{2}-\sigma\right)\theta}{\sin 2\theta} = \frac{1}{2}\frac{\cos 2\sigma\theta}{\cos\theta} - \frac{1}{2}\frac{\sin 2\sigma\theta}{\sin\theta}$$
$$\beta(\sigma) = \frac{\sin 2\left(\frac{1}{2}+\sigma\right)\theta}{\sin 2\theta} = \frac{1}{2}\frac{\cos 2\sigma\theta}{\cos\theta} + \frac{1}{2}\frac{\sin 2\sigma\theta}{\sin\theta}$$

Ideally, we would like to represent $P(s)$ in terms of an affine combination of A, B, and V. To do so, we write $C = M - dN$ and $V = m + hN$ to obtain $C = \frac{1}{2}(A+B) + \frac{1-g^2}{2g}(M-V)$. We then have

$$\begin{aligned} P(s) &= \alpha A + \beta B - (\alpha+\beta-1)\left[\frac{1+g^2}{2g^2}\frac{1}{2}(A+B) - \frac{1-g^2}{2g^2}V\right] \\ &= \lambda_A A + \lambda_B B + \lambda_V V \end{aligned}$$

Collecting terms, we see that

$$\lambda_V = \frac{1-g^2}{2g^2}(\alpha+\beta-1)$$

and

$$\lambda_{A,B} = \frac{1}{2}\left[1 \pm (\alpha-\beta)\right] - \frac{1}{2}\lambda_V$$

Using the above expressions for $\alpha(\sigma)$ and $\beta(\sigma)$,

$$\lambda_V = \frac{4}{1+g^2}\frac{\sin\left(\frac{1}{2}+\sigma\right)\theta}{\sin\theta}\frac{\sin\left(\frac{1}{2}-\sigma\right)\theta}{\sin\theta}$$

$$\lambda_{A,B} = \frac{1}{2}\left[1 \mp \frac{\sin 2\sigma\theta}{\sin\theta}\right] - \frac{1}{2}\lambda_V$$

Therefore, accurate computation of these quantities relies on computing $\frac{\sin(\alpha x)}{\sin(x)}$ for $\alpha \in [0,1]$.

10.1 Tangent vector

A tangent vector can be obtained by differentiating: $T = P'(s)$. We expect substantial simplification if we express the result in terms of $B - M$ and $V - M$. Notice that $\lambda'_A + \frac{1}{2}\lambda'_V = -\left(\lambda'_B + \frac{1}{2}\lambda'_V\right)$. Substituting this in, we get

$$\begin{aligned} P'(s) &= -(\lambda'_V + \lambda'_B)[M - (B-M)] + \lambda'_B[M + (B-M)] + \lambda'_V[M + (V-M)] \\ &= (\lambda'_V + 2\lambda'_B)(B-M) + \lambda'_V(V-M) \end{aligned}$$

Computing the derivatives gives

$$(\lambda'_V + 2\lambda'_B) = \frac{d}{d\sigma}\frac{\sin 2\sigma\theta}{\sin\theta}$$

$$\lambda'_V = \frac{4}{1+g^2}\left[\frac{d}{d\sigma}\left\{\frac{\sin\left(\frac{1}{2}+\sigma\right)\theta}{\sin\theta}\right\}\frac{\sin\left(\frac{1}{2}-\sigma\right)\theta}{\sin\theta} + \frac{\sin\left(\frac{1}{2}+\sigma\right)\theta}{\sin\theta}\frac{d}{d\sigma}\left\{\frac{\sin\left(\frac{1}{2}-\sigma\right)\theta}{\sin\theta}\right\}\right]$$

where it is assumed that the evaluation of the derivative of $\frac{\sin(\alpha x)}{\sin(x)}$ is atomic.

10.2 Subdivision

Subdividing an arc in half is straightforward since the position of V can be determined robustly. The new bulge parameters g' of the each half must be related to the original g. Since the angle θ becomes halved, then

$$g = \tan\frac{\theta}{2} = \frac{2g'}{1-g'^2}$$

Thus any stable method for computing the roots of $g\,g'^2 + 2g' - g = 0$ may be used. There is a single root if $g = 0$ corresponding to a straight segment. Otherwise, there are two and the correct choice is the one with the same sign as g.

11 Ray intersection

Let us express the ray as $X = X_0 + \tau D$ where D is the direction vector, and τ is the ray parameter (which may extend $-\infty$ for a line). Notice that if the ray intersects with the arc exactly once, it must intersect the chord as well. If the ray intersects with the arc (at most) twice, then it does not intersect the chord, and must intersect each of the other two sides of the triangular hull. These simple tests allow us to only consider the two cases.

In general, the intersection points between a ray and a circle are computed using

$$\|X_0 + \tau D - C\|^2 = r^2$$

For sufficiently large g, this equation is satisfactory. We consider in the following only the case when g is small.

11.1 Single intersection

When the ray intersects the chord of the arc, there must always be one intersection point. Rather than compute the intersection point analytically, it is simpler to use an iterative subdivision. It is clear that A and B are on opposites sides of the ray. We simply determine on which side lies V and continue bisecting the arc by maintaining a pair of endpoints which straddle the ray. In this way, the arc length parameter s of the intersection point is determined one binary digit at a time.

11.2 Two intersections

When the ray does not intersect the chord, but does intersect the other two edges of the triangular hull of the arc, there may not necessarily be any intersection. In the small g case, the intersection problem may be very ill-conditioned; since the ray and the arc are nearly parallel. In this case, a bisection method is still useful, but potentially more costly in terms of stack space. The arc is repeated bisected until (a) the ray does not intersect the triangular hull at all, or (b) the ray intersects with the arc only once. This reduces the problem back to the single intersection case.

12 Extremal points

The task here is to determine the point X or points on the arc which locally maximize the value $(X - X_0) \cdot D$ for some given point X_0 and some unit vector direction D.

12.1 $X_0 = \infty$

When X_0 is at infinity, only the direction D matters. In this case, we separate out into cases where the extremum is within the arc, and the cases when it is an endpoint. The extremum is within the arc only when $D \cdot P'(0) > 0$ and $D \cdot P'(1) < 0$, otherwise, the extremum is an endpoint and can be computed by treating the arc as a straight segment. For a straight segment, the extremum is simply B when $D \cdot (B - A) > 0$, otherwise the extremum is A.

When the extremum is in the interior of the arc, then we must have $X = C + rD$. The case when g is large is simple. When g is small, then resolve $\hat{D} = \alpha \frac{B-A}{\|B-A\|} + \beta N$.

$$
\begin{aligned}
X = C + rD &= M - t\frac{1-g^2}{2g}N + t\frac{1+g^2}{2g}D \\
&= V - t\frac{1+g^2}{2g}N + t\frac{1+g^2}{2g}D \\
&= V - t\frac{1+g^2}{2g}N + t\frac{1+g^2}{2g}\left[\alpha\frac{B-A}{2t} + \beta N\right] \\
&= V + t\frac{1+g^2}{2g}\left[\alpha\frac{B-A}{2t} - \frac{1-\beta}{gt}\left(V - \frac{1}{2}A - \frac{1}{2}B\right)\right]
\end{aligned}
$$

Now let us assume that we can compute $\alpha' = \frac{\alpha}{\sin\theta}$ and $\beta' = \frac{1-\beta}{g\sin\theta}$.

$$
\begin{aligned}
X &= V + \left[\alpha'\frac{B-A}{2} - \beta'\left(V - \frac{1}{2}A - \frac{1}{2}B\right)\right] \\
&= \lambda_A A + \lambda_B B + \lambda_V V
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_{A,B} &= \frac{1}{2}\beta' \mp \frac{1}{2}\alpha' \\
\lambda_V &= 1 - \beta'
\end{aligned}
$$

Computing α' and β' is then the primary difficulty.

We know that $\alpha \sim O(g)$ while $\beta \sim 1$ to lowest order in g. Therefore, we assume that we can compute

$$
\alpha = \frac{D \cdot (B-A)}{\|D\|\,\|B-A\|} \qquad \alpha' = \alpha\frac{1+g^2}{2g}
$$

We also know that $-1 \le \alpha' \le 1$. From this, $\beta = \sqrt{1-\alpha^2}$, and

$$
\beta' = \frac{1-\sqrt{1-\alpha^2}}{\alpha^2}\,\frac{2\alpha'^2}{1+g^2}
$$

where the first quantity is computed atomically.

13 Arc polygons

13.1 Polygon offset

The outward offset shape of any convex arc spline is another arc spline. In the case of a polygon, the outward offset shape is particularly easy to compute. Let P_i be the vertices of the polygon for $i = 0, \ldots, n-1$ arranged in CCW order. There are $2n$ vertices in the offset, corresponding to the endpoints of the offsets of each segment. The only difficulty is in determining the bulge parameters of the curved segments for each vertex. For a non-degenerate polygon, all the interior angles of its vertices are in the open interval $(0, \pi)$. Therefore the bulge parameters are in the open interval $(0, 1)$.

Let us assume that for each arc, we know A and B and have the outward unit normal vectors at each endpoint N_A and N_B. Let $\alpha = N_A \times N_B = \sin 2\theta$, and $\beta = N_A \cdot N_B = \cos 2\theta$, we can form the system of equations:

$$\alpha = 2 \sin\theta \cos\theta$$
$$\beta = 2\cos^2\theta - 1$$

Combining these gives

$$\alpha g^2 + 2(1+\beta)g - \alpha = 0$$

The product of the roots is again -1, so the two possibilities represent complement arcs on the parent circle. Note that in this derivation, we could have use oriented tangent vectors instead of normals.