On scattering matrices and the Redheffer star product

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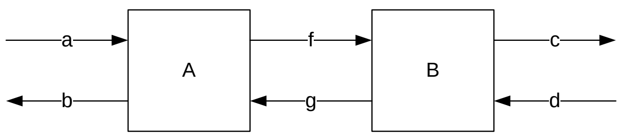
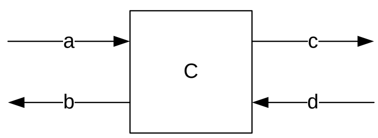
Abstract

Some useful properties about the star product.

1 Introduction

The star product is used for computing the scattering matrix from the two scattering matrices of two combined subsystems. Suppose we have two two-port systems with inputs and outputs





We can write the scattering matrices of each subsystem as

$$\begin{bmatrix} f \\ b \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} a \\ g \end{bmatrix} \qquad \begin{bmatrix} c \\ g \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} f \\ d \end{bmatrix} \tag{1}$$

and that of the combined system as

$$\begin{bmatrix} c \\ b \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} a \\ d \end{bmatrix} \tag{2}$$

There are two equations involving both f and g:

$$f = A_{11}a + A_{12}g \qquad g = B_{21}f + B_{22}d \tag{3}$$

Eliminating either f or g,

$$f = (I - A_{12}B_{21})^{-1}(A_{11}a + A_{12}B_{22}d) \tag{4}$$

$$g = (I - B_{21}A_{12})^{-1}(B_{21}A_{11}a + B_{22}d) \tag{5}$$

From these, we get that

$$C_{11} = B_{11}(I - A_{12}B_{21})^{-1}A_{11} \tag{6}$$
$$C_{12} = B_{11}(I - A_{12}B_{21})^{-1}A_{12}B_{22} + B_{12} \tag{7}$$
$$C_{21} = A_{21} + A_{22}(I - B_{21}A_{12})^{-1}B_{21}A_{11} \tag{8}$$
$$C_{22} = A_{22}(I - B_{21}A_{12})^{-1}B_{22} \tag{9}$$

Note the agreement between last and first subscripts of adjacent symbols. We will write this relationship as $C = A \star B$. Note that $(I - AB)^{-1}A = A(I - BA)^{-1}$, so we can rewrite the middle two equations above in another way.

2 Converting transfer to scattering matrices

The formulas for converting transfer matrices into scattering matrices has a number of useful properties. Denote by the hat operation:

$$\hat{S} = \begin{bmatrix} S_{11}^{-1} & -S_{11}^{-1}S_{12} \\ S_{21}S_{11}^{-1} & S_{22} - S_{21}S_{11}^{-1}S_{12} \end{bmatrix} \tag{10}$$

Given a transfer matrix T defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \tag{11}$$

the corresponding scattering matrix for the system is $S = \hat{T}$. The hat operation is what converts from scattering to transfer matrices, and thus converts from star products to ordinary matrix products. It was originally called the "linearization" operation by Redheffer. We will assume in the rest of this section that all the relevant inverses exist.

We have by block-wise inversion that

$$S^{-1} = \begin{bmatrix} S_{11}^{-1} + S_{11}^{-1}S_{12}\left(S_{22} - S_{21}S_{11}^{-1}S_{12}\right)^{-1}S_{21}S_{11}^{-1} & -S_{11}^{-1}S_{12}\left(S_{22} - S_{21}S_{11}^{-1}S_{12}\right)^{-1} \\ -\left(S_{22} - S_{21}S_{11}^{-1}S_{12}\right)^{-1}S_{21}S_{11}^{-1} & \left(S_{22} - S_{21}S_{11}^{-1}S_{12}\right)^{-1} \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} \left(S_{11} - S_{12}S_{22}^{-1}S_{21}\right)^{-1} & -\left(S_{11} - S_{12}S_{22}^{-1}S_{21}\right)^{-1}S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21}\left(S_{11} - S_{12}S_{22}^{-1}S_{21}\right)^{-1} & S_{22}^{-1} + S_{22}^{-1}S_{21}\left(S_{11} - S_{12}S_{22}^{-1}S_{21}\right)^{-1}S_{12}S_{22}^{-1} \end{bmatrix} \tag{13}$$

From this we get

$$\widehat{S^{-1}} = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & -S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} = \hat{S}^{-1} \tag{14}$$

and so therefore the hat operation commutes with inversion.

We can also verify that the hat operation is its own inverse:

$$\hat{\hat{S}} = S \tag{15}$$

Therefore, hat and inverse commute and are both their own inverses. We have the basic identities

$$\widehat{A \star B} = \hat{A}\hat{B} \tag{16}$$

3 Dissipation

For the scattering matrices A and B

$$\begin{bmatrix} f \\ b \end{bmatrix} = A \begin{bmatrix} a \\ g \end{bmatrix} \qquad \begin{bmatrix} c \\ g \end{bmatrix} = B \begin{bmatrix} f \\ d \end{bmatrix} \tag{17}$$

we call them dissipative if

$$\|f\|^2 + \|b\|^2 \leq \|a\|^2 + \|g\|^2 \qquad \|c\|^2 + \|g\|^2 \leq \|f\|^2 + \|d\|^2 \tag{18}$$

It is simple to see that the dissipative property is preserved by the star product. If

$$\begin{bmatrix} c \\ b \end{bmatrix} = (A \star B) \begin{bmatrix} a \\ d \end{bmatrix} \tag{19}$$

we have by adding the above inequalities

$$\|c\|^2 + \|b\|^2 \leq \|a\|^2 + \|d\|^2 \tag{20}$$

This can be easily extended to other norms and for equality. This means that the star product of two dissipative matrices is also dissipative. This also proves that the star product of two unitary matrices is unitary, and that the star product of two column-wise stochastic matrices is column-wise stochastic.