Fourier Transforms of Polyhedra

Victor Liu

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1 Notation

We define the Fourier transform by

$$\mathcal{F}\left\{g\right\}(\mathbf{k}) = \int g\left(\mathbf{r}\right) e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r} \tag{1}$$

where $\mathbf{k}$ is the wavevector, which is 2π times the spatial frequency vector. If this definition is the inverse transform of your convention, then conjugate all the results ($i \to -i$).

2 2D Polygon

We will largely present the derivation given in [1]. Define the indicator function for a region R as

$$I(\mathbf{r}) = \begin{cases} 1 & \mathbf{r} \in R \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The two dimensional Fourier transform of a region is then

$$S(\mathbf{k}) = \iint I(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r} = \iint_R e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r} \tag{3}$$

We assume the polygon has N vertices v_i for $i = 0, \ldots, N-1$ in counter-clockwise orientation. We understand that $\mathbf{v}_N = \mathbf{v}_0$ and that cross products between two in-plane vectors produces a scalar (the z component only). Define the vector field

$$\mathbf{F} = \begin{cases} \frac{1}{2}\hat{\mathbf{z}} \times \mathbf{r} & \mathbf{k} = 0 \\ \frac{ie^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2}\mathbf{k} \times \hat{\mathbf{z}} & \text{otherwise} \end{cases} \tag{4}$$

By Stokes theorem,

$$\iint_R \nabla \times \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} \tag{5}$$

we have that

$$\begin{aligned}
S(0) &= \hat{\mathbf{z}} \cdot \iint_R \frac{1}{2} \nabla \times \hat{\mathbf{z}} \times \mathbf{r} \, d\mathbf{r} = \frac{1}{2} \iint_R \nabla \mathbf{r} \, d\mathbf{r} = \iint_R d\mathbf{r} \\
&= \sum_{i=0}^{N-1} \frac{1}{2} \int_{-1}^{1} (\hat{\mathbf{z}} \times \mathbf{r}) \cdot (\mathbf{v}_{i+1} - \mathbf{v}_i) \, d\mathbf{r} \\
&= \frac{1}{2} \sum_{i=0}^{N-1} [(\mathbf{v}_{i+1} - \mathbf{v}_i) \times \hat{\mathbf{z}}] \cdot \int_{-1}^{1} \mathbf{r} \, d\mathbf{r}
\end{aligned}$$

where the integral is over the i-th edge. We can now parameterize the i-th edge by

$$\mathbf{r}(t) = \frac{1}{2} [(1-t)\mathbf{v}_i + (1+t)\mathbf{v}_{i+1}] \qquad -1 \leq t \leq 1 \tag{6}$$

Then we have

$$\begin{aligned}
S(0) &= \frac{1}{2} \sum_{i=0}^{N-1} [(\mathbf{v}_{i+1} - \mathbf{v}_i) \times \hat{\mathbf{z}}] \cdot \frac{1}{2} (\mathbf{v}_i + \mathbf{v}_{i+1}) \\
&= \frac{1}{2} \sum_{i=0}^{N-1} \hat{\mathbf{z}} \cdot (\mathbf{v}_i \times \mathbf{v}_{i+1})
\end{aligned} \tag{7}$$

which is the standard formula for the area of a polygon. Evaluating at $k \neq 0$, we first verify that the chosen field $\mathbf{F}$ is correct:

$$\nabla \times \mathbf{F} = i\mathbf{k} \times \mathbf{F} = -\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2} \mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{z}}) = -\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2} \hat{\mathbf{z}} (-\mathbf{k} \cdot \mathbf{k}) = \hat{\mathbf{z}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

Now applying Stokes' theorem gives

$$\begin{aligned}
S(\mathbf{k}) &= \frac{1}{2} \sum_{i=0}^{N-1} \int_{-1}^{1} \mathbf{F} \cdot (\mathbf{v}_{i+1} - \mathbf{v}_i) \, d\mathbf{r} \\
&= \frac{i}{2|\mathbf{k}|^2} \sum_{i=0}^{N-1} \int_{-1}^{1} e^{i\mathbf{k}\cdot\mathbf{r}(t)} (\mathbf{k} \times \hat{\mathbf{z}}) \cdot (\mathbf{v}_{i+1} - \mathbf{v}_i) \, dt \\
&= \frac{i}{2|\mathbf{k}|^2} \sum_{i=0}^{N-1} (\mathbf{k} \times \hat{\mathbf{z}}) \cdot (\mathbf{v}_{i+1} - \mathbf{v}_i) \, e^{i\mathbf{k}\cdot\mathbf{r}(0)} \int_{-1}^{1} e^{i\mathbf{k}\cdot(\mathbf{v}_{i+1} - \mathbf{v}_i)/2} \, dt \\
&= \frac{i}{|\mathbf{k}|^2} \sum_{i=0}^{N-1} \hat{\mathbf{z}} \cdot [(\mathbf{v}_{i+1} - \mathbf{v}_i) \times \mathbf{k}] \, e^{i\mathbf{k}\cdot(\mathbf{v}_{i+1} + \mathbf{v}_i)/2} \mathrm{sinc} \left(\frac{\mathbf{k} \cdot (\mathbf{v}_{i+1} - \mathbf{v}_i)}{2} \right)
\end{aligned} \tag{8}$$

where $\mathrm{sinc}(x) = \sin(x)/x$.

3 3D Polyhedron

We assume the polyhedron is specified in boundary representation by a facet mesh with N facets. Let T_i be the i-th facet for $i = 0, \ldots, N-1$. Let $|T_i|$ be the area of T_i, $\mathbf{n}_i$ be the outward unit normal vector of T_i, and $\mathbf{c}_i$ be the barycenter of T_i. Now instead of applying Stokes' theorem, we use the divergence theorem. Now we choose

$$\mathbf{F} = \begin{cases} \frac{1}{3}\mathbf{r} & \mathbf{k} = 0 \\ -\frac{ie^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2}\mathbf{k} & \text{otherwise} \end{cases} \tag{9}$$

We first derive the volume formula:

$$\begin{aligned} S(0) &= \hat{\mathbf{z}} \cdot \iiint_R \frac{1}{3} \nabla \cdot \mathbf{r}\, d\mathbf{r} = \iiint_R d\mathbf{r} \\ &= \frac{1}{3} \sum_{i=0}^{N-1} \iint_{T_i} \mathbf{r} \cdot \mathbf{n}_i \, d\mathbf{r} \\ &= \frac{1}{3} \sum_{i=0}^{N-1} |T_i| \, \mathbf{c}_i \cdot \mathbf{n}_i \end{aligned} \tag{10}$$

which recovers the standard volume for a polyhedron given in the boundary representation. In the above formula $\mathbf{c}_i$ can be replaced by any point on the facet. Evaluating at $k \neq 0$, we again verify that the chosen field $\mathbf{F}$ is correct:

$$\nabla \cdot \mathbf{F} = i\mathbf{k} \cdot \mathbf{F} = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2} \mathbf{k} \cdot \mathbf{k} = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Now applying the divergence theorem gives

$$\begin{aligned} S(\mathbf{k}) &= \frac{1}{3} \sum_{i=0}^{N-1} \iint_{T_i} \mathbf{F} \cdot \mathbf{n}_i \, d\mathbf{r} \\ &= -\frac{i}{3|\mathbf{k}|^2} \sum_{i=0}^{N-1} (\mathbf{k} \cdot \mathbf{n}_i) \iint_{T_i} e^{i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{r} \end{aligned}$$

The remaining integral can be computed using the polygon formula by projecting $\mathbf{k}$ into the facet plane:

$$S(\mathbf{k}) = -\frac{i}{3|\mathbf{k}|^2} \sum_{i=0}^{N-1} (\mathbf{k} \cdot \mathbf{n}_i) \, e^{i\mathbf{k}\cdot\mathbf{n}_i} S^{(i)} (\mathbf{k} - (\mathbf{k} \cdot \mathbf{n}_i) \, \mathbf{n}_i) \tag{11}$$

where

$$S^{(i)}(\mathbf{k}) = \frac{i}{|\mathbf{k}|^2} \sum_{i=0}^{N_i - 1} \mathbf{n}_i \cdot [(\mathbf{v}_{i+1} - \mathbf{v}_i) \times \mathbf{k}] \, e^{i\mathbf{k}\cdot(\mathbf{v}_{i+1}+\mathbf{v}_i)/2} \operatorname{sinc}\left(\frac{\mathbf{k} \cdot (\mathbf{v}_{i+1} - \mathbf{v}_i)}{2} \right) \tag{12}$$

where facet T_i has N_i vertices ordered in counter-clockwise orientation as $\mathbf{v}_i$ for $i = 0, \ldots, N_i - 1$.

References

[1] Kim McInturff and Peter S. Simon, *The Fourier transform of linearly varying functions with polygonal support*, IEEE Transactions on Antennas and Propagation vol. 39, no. 9 (1991).